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CURVES IN SPACE, CURVATURE, TORSION AND FRENET'S FORMULAE

Space curves

A curve in space can be represented in any one of the following ways,

(1) Let $f(x, y, z) = 0,\ \phi(x,y,z) = 0$ ... (1)

represent two surfaces. Then these two equations together will represent the curve of intersection of these surfaces. This curve will be called a plane curve if it lies in a plane, otherwise it is called a skew, twisted or tortuous curve.

(2) Let the co-ordinates of any point on a space curve be represented by the equations of the form :

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t) \qquad \ldots (2)$$

where f_1, f_2, f_3 are real valued functions of a single real variable t, ranging over a set of values $a \le t \le b$. The equations in (2) are called parametric equations of the space curve.

(3) Let $\bar{r}$ be the position vector of a point P on the space curve whose Cartesian coordinates are (x, y, z). Then

$$\bar{r} = xi + yj + zk$$

$$= f_1(t)i + f_2(t)j + f_3(t)k$$

Now, we define a curve in space as the locus of a point whose position vector $\bar{r}$ with respect to a fixed origin, may be expressed as a function of a single variable parameter t.

1. To find unit tangent vector to a curve

Let the equation of the curve be

$$\bar{r} = \bar{r}(t) \qquad \ldots (1)$$

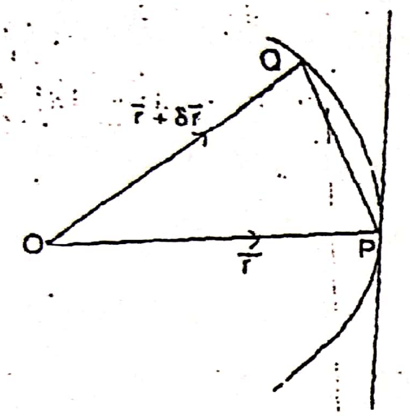
Let P and Q be two neighbouring points whose position vectors are $\bar{r}$ and $\bar{r} + \delta\bar{r}$ referred to 'O' as origin. Let the corresponding parameters be t and $t + \delta t$.

Then [illegible]

$\delta\bar{r} = \overline{PQ} = \overline{OQ} - \overline{OP}$

$= \bar{r}(t+\delta t) - \bar{r}(t)$

$$\therefore \frac{\delta\bar{r}}{\delta t} = \frac{\bar{r}(t+\delta t) - \bar{r}(t)}{\delta t}$$



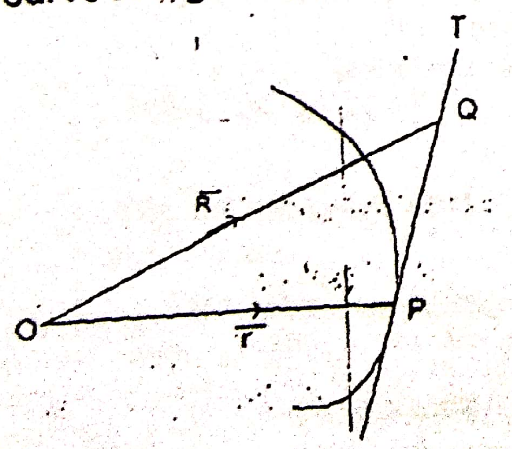
Taking the limit as $Q \to P$ (i.e.,) $\delta t \to 0$, $\overline{PQ}$ will tend to be a tangent at P.

$\therefore$ We conclude that $\dot{\bar{r}} = \frac{d\bar{r}}{dt}$ will be a vector parallel to the tangent at P. If $\bar{t}$ denotes the unit tangent vector, then

$$\bar{t} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{d\bar{r}}{ds} = \bar{r}'. \text{ Thus } \bar{t} = \bar{r}'$$

Note: We use dashes to denote the differentiation with respect to arc lengths s and dots to denote the differentiation with respect to any other parameter t.

2. To find the equation of tangent line to a curve at a given point.



Let $\bar{r} = \bar{r}(t)$ be the parametric equation to a curve and P be any point on it, whose position vector is $\bar{r}$.

A unit tangent at P is denoted by

$$\bar{t} = \frac{d\bar{r}}{ds} = \bar{r}'$$

Let $\bar{R}$ be the position vector of any point Q on the tangent line. We have

$$\overline{PQ} = c\bar{t} \text{ where } c = |\overline{PQ}|$$

$\therefore$ The equation of the tangent line at P is given by

$$\bar{R} = \bar{r} + c\bar{t}$$

or $\bar{R} = \bar{r} + c\bar{r}'$, where c is a parameter. ... (1)

Again, we know that $\dot{\bar{r}} = \frac{d\bar{r}}{dt}$ is also a vector along the tangent P. Therefore, the equation of the tangent line can also be written as

$$\bar{R} = \bar{r} + \lambda\dot{\bar{r}} \quad \ldots (2)$$

where λ is a scalar.

Cor: To find the tangent line in Cartesians

Let $\bar{r} = xi + yj + zk$, $\bar{R} = Xi + Yj + Zk$

then $\bar{r}' = \frac{d\bar{r}}{ds} = \frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k$

$\dot{\bar{r}} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$

Substituting for $\bar{R}$, $\bar{r}$ and $\bar{r}'$ in (1),

$$Xi + Yj + Zk = xi + yj + zk + c\left[\frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k\right]$$

Equating the coefficients of i, j, k, we get

$$\frac{X - x}{\frac{dx}{ds}} = \frac{Y - y}{\frac{dy}{ds}} = \frac{Z - z}{\frac{dz}{ds}} = c$$

This represents the equation of the tangent line at (x, y, z) and $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction cosines.

Again substituting the values of $\bar{R}, \bar{r}$ and $\bar{t}$ in (2).

$$Xi + Yj + Zk = xi + yj + zk + \lambda \left[\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k\right]$$

i.e., $$\frac{X - x}{\frac{dx}{dt}} = \frac{Y - y}{\frac{dy}{dt}} = \frac{Z - z}{\frac{dz}{dt}} = \lambda$$

This equation also represents the tangent line at P(x, y, z) but $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are direction ratios and not direction cosines.

Osculating plane or plane of curvature.

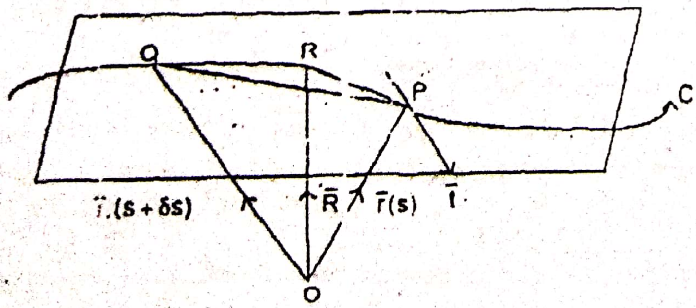
Definition: Let Q and R be any two points on the curve which are very close to the point P on the curve. Then the limiting position of the plane PQR as the points Q and R tend to P is called the Osculating plane at the point P.

3. To find the equation of the osculating plane

Let the curve C be given by

$$\bar{r} = \bar{r}(s)$$

where the parameter s is the length of the arc of curve measured from a fixed point on the curve.



Let the points P and Q correspond to the parameters s and s + δs so that their position vectors are $\bar{r}(s)$ and $\bar{r}(s + \delta s)$.

Let $\bar{R}$ be the position vector of R.

We have $\overline{PQ} = \overline{OQ} - \overline{OP}$

$$= \bar{r}(s + \delta s) - \bar{r}(s)$$

$$\overline{PR} = \bar{R} - \bar{r}(s)$$

and $$\bar{t} = \frac{d\bar{r}}{ds} = \bar{r}'$$

The three vectors $\overline{PQ}, \overline{PR}$ and $\bar{t}$ are coplanar. Hence their scalar triple product is zero.

i.e., $\left[\overline{PR}, \bar{t}, \overline{PQ}\right] = 0$

i.e., $\left[\bar{R} - \bar{r}(s), \bar{t}, \bar{r}(s+\delta s) - \bar{r}(s)\right] = 0$

i.e., $\bar{R} - \bar{r}(s)\,\bar{t} \times [\,\bar{r}(s + \delta s) - \bar{r}(s)] = 0$ ...(1)

Expanding $\bar{r}(s+\delta s)$ by Taylor's Theorem,

$$\bar{r}(s+\delta s) = \bar{r}(s) + \delta s\,\bar{r}'(s) + \frac{(\delta s)^2}{\lfloor 2}\bar{r}''(s) + (\delta s)^3$$

$$\therefore \bar{r}(s+\delta s) - \bar{r}(s) = \delta s\,\bar{r}'(s) + \frac{(\delta s)^2}{\lfloor 2}\bar{r}''(s) + (\delta s)^3$$

Substituting in (1)

$$\{\bar{R} - \bar{r}(s)\} \cdot \bar{t} \times \left[\delta s\,\bar{r}'(s) + \frac{\delta s^2}{\lfloor 2}\bar{r}''(s) + (\delta s)^3\right] = 0$$

Since $\bar{t}\times\bar{r}'(s)=\bar{r}'(s)\times\bar{r}'(s)=0$ the above equation becomes

$$\therefore\left[\bar{R}-\bar{r}(s)\right].\bar{t}\times\left[\frac{(\delta s)^2}{\lfloor 2}\bar{r}''(s)+(\delta s)^3\right]=0$$

i.e., $\left[\bar{r}-\bar{r}(s)\right].\bar{r}'(s)\times\left[\bar{r}''(s)+(\delta s)\right]=0$

The above plane becomes an osculating plane as $Q\to P$ i.e., as $\delta x\to 0$

The equation of the osculating plane is

$$\left[\bar{R}-\bar{r}(s).\bar{r}'(s)\times\bar{r}''(s)\right]=0$$

(i.e.,) $\left[\bar{R}-\bar{r}(s),\bar{r}'(s),\bar{r}''(s)\right]=0$

Cor: (1) Suppose the length of arc is measured from the point P. Then s = 0. The equation becomes

$$\left[\bar{R}-\bar{r}(0),\bar{r}'(0),\bar{r}''(0)\right]=0$$

(2) If t is any other parameter,

$$\bar{r}'=\frac{d\bar{r}}{ds}=\frac{d\bar{r}}{dt}\frac{dt}{ds}=\frac{\dot{\bar{r}}}{\dot{s}}$$

$$\bar{r}''=\frac{d}{ds}\left(\frac{\dot{\bar{r}}}{\dot{s}}\right)=\frac{d}{dt}\left(\frac{\dot{\bar{r}}}{\dot{s}}\right)\frac{dt}{ds}$$

$$=\frac{1}{\dot{s}}\frac{d}{dt}\left(\frac{\dot{\bar{r}}}{\dot{s}}\right)$$

$$=\frac{1}{\dot{s}}\left[\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s}}{(\dot{s})^2}\right]=\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s}}{(\dot{s})^3}$$

The equation becomes,

$$\left[\bar{R}-\bar{r},\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s}}{(\dot{s})^3}\right]=0$$

or $\left[\bar{R}-\bar{r},\dot{\bar{r}},\ddot{\bar{r}}\right]=0$, since $\dot{\bar{r}}\cdot\dot{\bar{r}}=0$

(3) Equation in Cartesians:

Let $\bar{R}=Xi+Yj+Zk$ and $\bar{r}=xi+yj+zk$. Then the equation of the osculating plane wil be

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}=0$$

4. To find the equation of the normal plane at a point P.

The plane through P perpendicular to the tangent line at P is called the normal plane at P.

Let the position vector of P be $\bar{r}$. Take a point Q on the normal plane and let the position vector of Q be $\bar{R}$. Then

$$\overline{PQ}=\overline{OQ}-\overline{OP}=\bar{R}-\bar{r}$$

$\overline{PQ}$ is perpendicular to $\frac{d\bar{r}}{dt}$. Therefore the equation of the normal plane is

$$(\bar{R}-\bar{r})\cdot\frac{d\bar{r}}{dt}=0$$

Corollary: The normal plane is perpendicular to the osculating plane.

From the equation of the normal plane, we can say that it is perpendicular to $\frac{d\bar{r}}{dt}$.

From the equation of the osculating plane

$\left[\bar{R}-\bar{r},\dot{\bar{r}},\ddot{\bar{r}}\right]=0$, we infer that the osculating plane is perpendicular to $\dot{\bar{r}}\times\ddot{\bar{r}}$

$$\text{Since }\frac{d\bar{r}}{dt}\cdot\left(\dot{\bar{r}}\times\ddot{\bar{r}}\right)=\frac{d\bar{r}}{dt}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)=0$$

We conclude that the normal plane is perpendicualr to the osculating plane.

Example 1

Find the equation of the osculating plane at the point on the helix

$x=a\cos t,\ y=a\sin t,\ z=ct$

The equation of the osculating plane is,

$$\begin{vmatrix} X-a\cos t & Y-a\sin t & Z-ct \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix}=0$$

$(X - a\cos t)(ca \sin t) - (Y - a \sin t)$

$(ca \cos t) + (Z - ct)(a^2) = 0$

i.e., $c \sin t\, X - ca \sin t \cos t - cY \cos t + ac \sin t \cos t + aZ - act = 0$

$$c(X \sin t - Y \cos t) + a(Z - ct) = 0$$

Normal lines at a point

Any line perpendicular to the tangent at a point P is called a normal at P. The normal plane at a point is the locus of the normal through the point. Of all the normals at a point, two normals known as Principal normal and Binormal are specially important.

Principal normal

The principal normal at any point P of a given curve C is defined as the normal which lies in the osculating plane at P. i.e., the principal normal is the line of intersection of the osculating plane and the normal plane at the point.

Binormal

The normal which is perpendicular to the osculating plane at a point is called the Binormal. Obviously this is perpendicular to the principal normal.

The unit vectors along the principal normal and binormal are denoted by $\bar{n}$ and $\bar{b}$ respectively.

To find the directions of Principal normal and Binormal.

The equation of the osculating plane is

$$\left[\bar{R} - \bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}\right] = 0$$

$$\text{i.e., } \left[(\bar{R} - \bar{r}) \cdot \left(\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right)\right] = 0$$

Since the binormal is perpendicular to the osculating plane, it is parallel to $\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = 0.$

Since the principal normal is perpendicular to the tangent and also to the binormal, it is parallel to

$$\left(\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right) \times \left(\frac{d\bar{r}}{dt}\right)$$

$$\text{i.e., } \left(\frac{d\bar{r}}{dt}\right)^2 \frac{d^2\bar{r}}{dt^2} - \left(\frac{d\bar{r}}{dt} \cdot \frac{d^2\bar{r}}{dt^2}\right) \frac{d\bar{r}}{dt}$$

If the length of arc s is taken as the parameter, then $\frac{d\bar{r}}{ds}$ is the unit tangent vector at the point

$$\text{i.e., } \left|\frac{d\bar{r}}{ds}\right| = 1$$

$$\frac{d\bar{r}}{ds} \cdot \frac{d\bar{r}}{ds} = 1$$

Differentiating, we get

$$2\frac{d\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds^2} = 0 \qquad \dots (1)$$

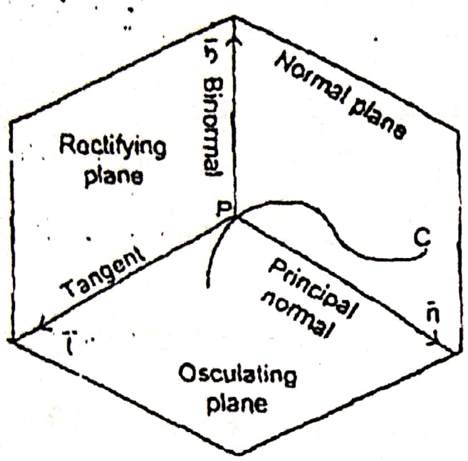
This relation shows that $\frac{d^2\bar{r}}{ds^2}$ is perpendicular to the tangent and hence it is parallel to the normal. Further, from the equation of the osculating plane $(\bar{R} - \bar{r}) \cdot \left(\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right) = 0$, it is clear that $\frac{d^2\bar{r}}{ds^2}$ lies in the osculating plane and both these results prove that $\frac{d^2\bar{r}}{ds^2}$ is parallel to the principal normal. Since binormal is perpendicular to the osculating plane, it is parallel to $\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}$.

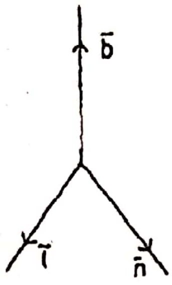
Orthonormal triad of vectors

We know that the principal normal and Binormal are perpendicular to each other and both these lines being normals are perpendicular to the tangent. Hence the three vectors $\bar{t}$, $\bar{n}$, $\bar{b}$ form a triad of three mutually perpendicular unit vectors such that $\bar{t}$, $\bar{n}$, $\bar{b}$ form a right handed orthogonal system of axes.

Thus $\bar{t} \times \bar{n} = \bar{b}, \bar{n} \times \bar{b} = \bar{t}, \bar{b} \times \bar{t} = \bar{n}$

and $\bar{t} \cdot \bar{n} = \bar{n} \cdot \bar{b} \cdot \bar{t} = 0$





Thus at each point on the curve, there is a triad of orthogonal unit vectors which determine three mutually perpendicular planes. The planes are

(1) The osculating plane containing $\bar{n}$ and $\bar{t}$ and its equation is $(\bar{R} - \bar{r}) \cdot \bar{b} = 0$

(2) The normal plane containing $\bar{n}$ and $\bar{b}$ and its equation is $(\bar{R} - \bar{r}) \cdot \bar{t} = 0$

(3) The rectifying plane containing $\bar{b}$ and $\bar{t}$ whose equation is $(\bar{R} - \bar{r}) \cdot \bar{n} = 0$

Curvature

Definition: The rate of change of the direction of tangent with respect to the arc length 's', as the point P moves along the curve is called **curvature vector** of the curve. Its magnitude is denoted by k (kappa) which is called the **curvature at P.** The reciprocal of the curvature is called the **radius of curvature** and is denoted by ρ

$$\therefore k = \left|\frac{d\bar{t}}{ds}\right| = \left|\frac{d^2\bar{r}}{ds^2}\right| \text{ and } \rho = \frac{1}{k}$$

Torsion: The rate of change of the direction of binormal with respect to the arc length 's' as the point P moves along the curve is called the **torsion vector** of the curve, whose magnitude is denoted by τ (tau) which is called the **torsion at P.** The reciprocal of the torsion is called the **radius of torsion** and is denoted by $\sigma = \frac{1}{\tau}$.

$$\therefore \tau = \left|\frac{d\bar{b}}{ds}\right|$$

Screw curvature: The rate of change of the direction of principal normal with respect to the arc length 's' as the point P moves along the curve is called the **screw curvature vector** and its magnitude is $\sqrt{k^2 + \tau^2}$.

$$\therefore \sqrt{k^2 + \tau^2} = \left|\frac{d\bar{n}}{ds}\right|$$

SERRET-FRENET FORMULAE

5. **The arc derivatives of the three unit vectors $\bar{t}, \bar{n}, \bar{b}$ are given by three relations which are known as Serret-Frenet formulae.**

(I.A.S. Paper II – 1980)

(1) $\bar{t}' = \frac{d\bar{t}}{ds} = k\bar{n}$ (2) $\frac{d\bar{b}}{ds} = \tau\bar{n}$

(3) $\bar{n}' = \frac{d\bar{n}}{ds} = \tau\bar{b} - k\bar{t}$

Proof

(1) Since $|\bar{t}| = \left|\frac{d\bar{r}}{ds}\right| = 1$, we have

$$\frac{d\bar{r}}{ds} \cdot \frac{d\bar{r}}{ds} = 1$$

Differentiating with respect to 's'

$$\frac{d\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds^2} = 0$$

This relation shows that $\frac{d^2\bar{r}}{ds^2}\left(= \frac{d\bar{t}}{ds}\right)$ is perpendicular to the tangent and hence it lies in the normal plane. From the equation of the osculating plane $(\bar{R}-\bar{r})\cdot\left(\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2}\right)=0$; we see that $\frac{d^2\bar{r}}{ds^2}\left(=\frac{d\bar{t}}{ds}\right)$ lies on the osculating plane, and hence it lies along the intersection of the normal plane and osculating plane. In otherwords, it lies along the principal normal and hence it is proportional to $\bar{n}$

$$\therefore \frac{d\bar{t}}{ds}=\pm\lambda\bar{n}$$

From the definition of curvature, we know that

$$\therefore \left|\frac{d\bar{t}}{ds}\right|=k \text{ and } |\bar{n}|=1$$

It follows that $\frac{d\bar{t}}{ds}=\pm k\bar{n}$

Now by convention, we take the positive sign

$$\therefore \frac{d\bar{t}}{ds}=k\bar{n}$$

(2) Since $\bar{b}$ is a unit vector

$$\bar{b}\cdot\bar{b}=1$$

Differentiating with respect to s,

$$\therefore \bar{b}\cdot\frac{d\bar{b}}{ds}=0$$

This shows that $\frac{d\bar{b}}{ds}$ is perpendicular to $\bar{b}$ and hence $\frac{d\bar{b}}{ds}$ lies in the osculating plane.

Also $\bar{b}\cdot\bar{t}=0$

Differentiating this equation with respect to s,

$$\bar{b}\cdot\frac{d\bar{t}}{ds}+\frac{d\bar{b}}{ds}\cdot\bar{t}=0$$

i.e., $\bar{b}\cdot k\bar{n}+\frac{d\bar{b}}{ds}\cdot\bar{t}=0$

i.e., $\frac{d\bar{b}}{ds}\cdot\bar{t}=0$ since $\bar{b}\cdot\bar{n}=0$

$\therefore \frac{d\bar{b}}{ds}$ is perpendicular to both $\bar{b}$ and $\bar{t}$. Hence it is perpendicualr to the plane containing $\bar{b}$ and $\bar{t}$.

Hence it lies along the principal normal $\bar{n}$ and is proportional to $\bar{n}$

$$\therefore \frac{d\bar{b}}{ds}=\pm\mu\bar{n}$$

From the definition of torsion

$$\left|\frac{d\bar{b}}{ds}\right|=\tau \text{ and } |\bar{n}|=1$$

$$\therefore \frac{d\bar{b}}{ds}=\pm\tau\bar{n}$$

By convention, we write

$$\frac{d\bar{b}}{ds}=-\tau\bar{n}$$

(3) We know that

$$\bar{n}=\bar{b}\times\bar{t}$$

Differentiating the above equation with respect to s

$$\frac{d\bar{n}}{ds}=\frac{d\bar{b}}{ds}\times\bar{t}+\bar{b}\times\frac{d\bar{t}}{ds}$$

$$=\frac{d\bar{b}}{ds}\times\bar{t}+\bar{b}\times k\bar{n}$$

$$=-\tau\bar{n}\times\bar{t}+\bar{b}\times k\bar{n}$$

Since $\bar{t}$, $\bar{n}$, $\bar{b}$ form a right handed orthogonal system

$$\bar{t}\times\bar{n}=\bar{b} \text{ and } \bar{n}\times\bar{b}=\bar{t}$$

$$\therefore \frac{d\bar{n}}{ds}=\tau\bar{b}-k\bar{t}$$

Remarks: Serret Frenet formula can be represented in the form of a matrix equation thus

$$\begin{vmatrix} \frac{d\bar{t}}{ds} \\ \frac{d\bar{n}}{ds} \\ \frac{d\bar{b}}{ds} \end{vmatrix} = \begin{vmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{vmatrix} \begin{vmatrix} \bar{t} \\ \bar{n} \\ \bar{b} \end{vmatrix}$$

6. The necessary and sufficient condition that a given curve is a plane curve is that $\tau = 0$ at all points.

Proof:

Necessary

If the given curve is a plane curve, the tangent and normal at all points lie in the plane of the curve. i.e., the plane of the curve is osculating plane at all points of the curve. Hence binormal $\bar{b}$ is the same at all points which mean that $\bar{b}$ is a constant vector both in magnitude and direction and hence,

$$\frac{d\bar{b}}{ds} = 0. \text{ (i.e.,) } -\tau\bar{n} = 0 \text{ or } \tau = 0$$

Sufficient

Suppose $\tau = 0$. Then $\frac{d\bar{b}}{ds} = 0$ and hence $\bar{b}$ is a constant vector. This means that the osculating plane is the same at all points of the curve. i.e., osculating plane contains the curve. Hence the curve is a plane curve.

7. The necessary and sufficient condition for the curve to be a plane curve is

$$\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right] = 0$$

Proof:

We have $\frac{d\bar{t}}{ds} = k\bar{n}$ by Frenet's formula

$$\therefore \frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2} = \bar{t} \times k\bar{n} = k(\bar{t} \times \bar{n}) = k\bar{b} \quad \ldots(1)$$

Differentiating with respect to s

$$\frac{d\bar{r}}{ds} \times \frac{d^3\bar{r}}{ds^3} + \frac{d^2\bar{r}}{ds^2} \times \frac{d^2\bar{r}}{ds^2} = \frac{dk}{ds}\bar{b} + k\frac{d\bar{b}}{ds}$$

$$\therefore \frac{d\bar{r}}{ds} \times \frac{d^3\bar{r}}{ds^3} + 0 = \frac{dk}{ds}\bar{b} + k(-\tau\bar{n})$$

$$\therefore \left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right] = \frac{d^2\bar{r}}{ds^2} \cdot \left(\frac{d^3\bar{r}}{ds^3} \times \frac{d\bar{r}}{ds}\right)$$

$$= -\frac{d^2\bar{r}}{ds^2} \cdot \left(\frac{d\bar{r}}{ds} \times \frac{d^3\bar{r}}{ds^3}\right)$$

$$= -\frac{d^2\bar{r}}{ds^2} \cdot \left(\frac{dk}{ds}\bar{b} - k\tau\bar{n}\right)$$

$$= -k\bar{n} \cdot \left(\frac{dk}{ds}\bar{b} - k\tau\bar{n}\right)$$

$$= k^2\tau \text{ since } \bar{n} \cdot \bar{b} = 0$$

$$\text{and } \bar{n} \cdot \bar{n} = 1$$

$$\therefore \left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right] = k^2\tau \quad \ldots(2)$$

If $\tau = 0$ then from the above theorem, the curve is a plane.

If $\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right] = 0$, then $k^2\tau = 0$.

Then either

$k = 0$ or $\tau = 0$ since $k \neq 0$; $\tau = 0$

Corollary

$$\frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \frac{d\bar{r}}{ds}\dot{s} \quad \ldots(3)$$

$$\frac{d^2\bar{r}}{dt^2} = \frac{d\bar{r}}{ds}\ddot{s} + \frac{d^2\bar{r}}{ds^2}\dot{s}^2 \quad \ldots(4)$$

$$\frac{d^3\bar{r}}{dt^3} = \frac{d\bar{r}}{ds}\dddot{s} + \frac{d^2\bar{r}}{ds^2}\dot{s}\ddot{s} + \frac{d^3\bar{r}}{ds^3}\dot{s}\dot{s}^2 + \frac{d^2\bar{r}}{ds^2}2\dot{s}\ddot{s}$$

$$= \dot{s}^3\frac{d^3\bar{r}}{ds^3} + 3\dot{s}\ddot{s}\frac{d^2\bar{r}}{ds^2} + \dddot{s}\frac{d\bar{r}}{ds}$$

$$\therefore \left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right] = \left[\frac{d\bar{r}}{ds}\dot{s}, \frac{d\bar{r}}{ds}\ddot{s} + \frac{d^2\bar{r}}{ds^2}\dot{s}^2, \dot{s}^3\frac{d^3\bar{r}}{ds^3} + 3\dot{s}\ddot{s}\frac{d^2\bar{r}}{ds^2} + \dddot{s}\frac{d\bar{r}}{ds}\right]$$

We know that the scalar triple product vanishes if two vectors are equal.

$$\therefore \left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right] = \left[\frac{d\bar{r}}{ds}\dot{s}, \frac{d^2\bar{r}}{ds^2}\dot{s}^2, \frac{d^3\bar{r}}{ds^3}\dot{s}^3\right]$$

$$= \dot{s}^6\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right].$$

Since $\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right] = 0$ is the necessary and sufficient condition for the curve to be a plane curve, it follows that $\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^3}, \frac{d^3\bar{r}}{dt^3}\right] = 0$ is also the necessary and sufficient condition for the curve to be a plane curve.

8. Expressions for curvature and torsion

$$(1)\ k = \left|\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right| = \frac{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|}{\left|\frac{d\bar{r}}{dt}\right|^3}$$

$$(2)\ \tau = \frac{\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right]}{\left|\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right|^2} = \frac{\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right]}{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|^2}$$

Proof:

We have from (1) of Article 7

$$\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2} = k\bar{b}$$

$$\therefore \left|\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right| = k$$

Again from (3) and (4) of Article 7

$$\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} = \frac{d\bar{r}}{ds}\dot{s} \times \left(\frac{d\bar{r}}{ds}\ddot{s} + \frac{d^2\bar{r}}{ds^2}\dot{s}^2\right)$$

$$= \dot{s}^3\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2} \text{ since } \frac{d\bar{r}}{ds} \times \frac{d\bar{r}}{ds} =$$

$$\therefore k = \left|\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right| = \frac{1}{\dot{s}^3}\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|$$

$$= \frac{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|}{\left|\frac{d\bar{r}}{dt}\right|^3} \text{ since } \left|\frac{d\bar{r}}{dt}\right| = \left|\frac{d\bar{r}}{ds}\frac{ds}{dt}\right|$$

$$= \left|\frac{d\bar{r}}{ds}\right|\dot{s} = \dot{s}$$

Again from (2) of Article 7,

$$k^2\tau = \left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right]$$

$$\therefore \tau = \frac{\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right]}{k^2} = \frac{\left[\frac{d\bar{r}}{ds}, \frac{d^2\bar{r}}{ds^2}, \frac{d^3\bar{r}}{ds^3}\right]}{\left|\frac{d\bar{r}}{ds} \times \frac{d^2\bar{r}}{ds^2}\right|^2}$$

τ is also equal to

$$\tau = \frac{\left[\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \frac{d^3\bar{r}}{dt^3}\right]}{\left|\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right|^2}$$

Example 2

Find $f(\theta)$ so that the curve

$x = a\cos\theta$, $y = a\sin\theta$, $z = f(\theta)$

determine a plane curve.

The required condition is

$$\left(\frac{d\bar{r}}{d\theta}, \frac{d^2\bar{r}}{d\theta^2}, \frac{d^3\bar{r}}{d\theta^3}\right) = 0$$

where $\bar{r} = [a\cos\theta, a\sin\theta, f(\theta)]$

$$\frac{d\bar{r}}{d\theta} = [-a\sin\theta, a\cos\theta, f'(\theta)]$$

$$\frac{d^2\bar{r}}{d\theta^2} = [-a\cos\theta, -a\sin\theta, f''(\theta)]$$

$$\frac{d^3\bar{r}}{d\theta^3} = [a\sin\theta, -a\cos\theta, f'''(\theta)]$$

$$\left[\frac{d\bar{r}}{d\theta}, \frac{d^2\bar{r}}{d\theta^2}, \frac{d^2\bar{r}}{d\theta^3}\right]$$

$$= \begin{vmatrix} -a\sin\theta & a\cos\theta & f'(\theta) \\ -a\cos\theta & -a\sin\theta & f''(\theta) \\ a\sin\theta & -a\cos\theta & f'''(\theta) \end{vmatrix} = 0$$

$$= \begin{vmatrix} 0 & 0 & f'(\theta)+f'''(\theta) \\ -a\cos\theta & -a\sin\theta & f''(\theta) \\ a\sin\theta & -a\cos\theta & f'''(\theta) \end{vmatrix} = 0$$

i.e., $f'(\theta) + f'''(\theta) = 0$

i.e., $\frac{df}{d\theta} + \frac{d^3f}{d\theta^3} = 0$

i.e., $(D^3 + D)f = 0$, where D stands for $\frac{d}{d\theta}$

The auxiliary equation is $m^3 + m = 0$ i.e., $m(m^2+1) = 0$

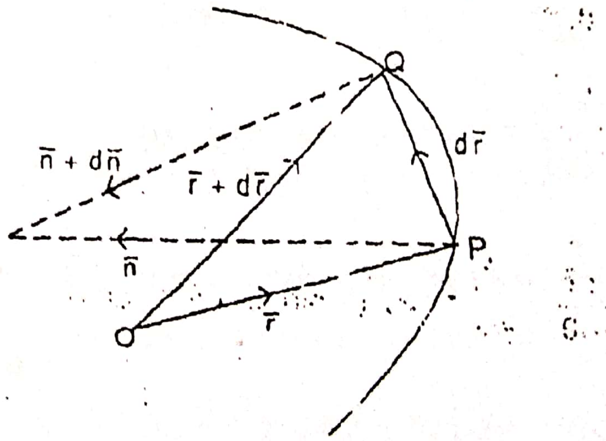
$\therefore m = 0, \pm i$

The general solution is $f(\theta) = A + B\cos\theta + C\sin\theta$

Where A, B, C are arbitrary constants.

Example 3

Prove that the principal normals at consecutive points do not intersect unless $\tau = 0$. [Paper II - 1983]



Let the position vectors of two consecutive points P and Q be $\bar{r}$ and $\bar{r} + d\bar{r}$.

Let $\bar{n}$ and $\bar{n} + d\bar{n}$ be the unit principal normals at these two points. Now the principal normals will intersect if the three vectors $d\bar{r}$, $\bar{n}$ and $\bar{n} + d\bar{n}$ are coplanar

i.e., if $[d\bar{r}, \bar{n}, \bar{n} + d\bar{n}] = 0$

or $[d\bar{r}, \bar{n}, d\bar{n}] = 0$

or $\left[\frac{d\bar{r}}{ds}, \bar{n}, \frac{d\bar{n}}{ds}\right] = 0$

or $[\bar{t}, \bar{n}, \tau\bar{b} - k\bar{t}] = 0$

or $\tau[\bar{t}, \bar{n}, \bar{b}] = 0$

$\therefore$ either $\tau = 0$ or $[\bar{t}, \bar{n}, \bar{b}]$

$= 0$, But $[\bar{t}, \bar{n}, \bar{b}] = 1 \neq 0$

$\therefore \tau = 0$

Example 4

Find the radius of curvature and torsion of the helix

$x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\tan\alpha$

$\bar{r} = (a\cos\theta, a\sin\theta, a\theta\tan\alpha)$

$$\frac{d\bar{r}}{d\theta} = (-a\sin\theta, a\cos\theta, a\tan\alpha)$$

$$\frac{d^2\bar{r}}{d\theta^2} = (-a\cos\theta, -a\sin\theta, 0)$$

$$\frac{d^3\bar{r}}{d\theta^3} = (a\sin\theta, -a\cos\theta, 0)$$

$$\left|\frac{d\bar{r}}{d\theta}\right| = a\left(\sin^2\theta + \cos^2\theta + \tan^2\alpha\right)^{\frac{1}{2}}$$

$$= a\sec\alpha$$

$$\left|\frac{d\bar{r}}{d\theta} \times \frac{d^2\bar{r}}{d\theta^2}\right| = a^2(-\sin\theta, \cos\theta, \tan\alpha)$$

$$\times(-\cos\theta, -\sin\theta, 0)$$

$$= a^2(\sin\theta\tan\alpha, -\cos\theta\tan\alpha, 1)$$

$$\left|\frac{d\bar{r}}{d\theta} \times \frac{d^2\bar{r}}{d\theta^2}\right| = a^2[\sin^2\theta\tan^2\alpha$$

$$+ \cos^2\theta\tan^2\alpha + 1]^{\frac{1}{2}}$$

$$= a^2\sec\alpha$$

$$\therefore k = \frac{\left|\frac{d\bar{r}}{d\theta} \times \frac{d^2\bar{r}}{d\theta^2}\right|}{\left|\frac{d\bar{r}}{d\theta}\right|^3} = \frac{a^2\sec\alpha}{a^3\sec^3\alpha} = \frac{\cos^2\alpha}{a}$$

$$\therefore \rho = \frac{1}{k} = a\sec^2\alpha$$

$$\tau = \frac{\left[\frac{d\bar{r}}{d\theta}, \frac{d^2\bar{r}}{d\theta^2}, \frac{d^3\bar{r}}{d\theta^3}\right]}{\left|\frac{d\bar{r}}{d\theta} \times \frac{d^2\bar{r}}{d\theta^2}\right|} = \frac{a^3\tan\alpha}{a^4\sec^2\alpha}$$

$$= \frac{1}{a}\sin\alpha\cos\alpha$$

$\therefore \sigma = a\sec\alpha\,\mathrm{cosec}\,\alpha$

Example 5

A curve is drawn on a cylinder of radius a and the cylinder is developed into a plane. If ρ is the radius of curvature of the curve and ρ_1 is the radius of curvature of the developed curve at the corresponding points, then

$$\frac{1}{\rho^2} - \frac{1}{\rho_1^2} = \frac{\sin^4\phi}{a^2}$$

where ϕ is the angle that the tangent to the curve makes with the generator of the cylinder through this point.

Any point on the cylinder is given by

$x = a\cos\theta,\ y = -a\sin\theta,\ z = z$

$\therefore \bar{r} = (a\cos\theta, -a\sin\theta, z)$

$$\frac{d\bar{r}}{d\theta} = \left(-a\sin\theta, -a\cos\theta, \frac{dz}{d\theta}\right)$$

$$\frac{d^2\bar{r}}{d\theta^2} = \left(-a\cos\theta, a\sin\theta, \frac{d^2z}{d\theta^2}\right)$$

$$\frac{d\bar{r}}{d\theta}\times\frac{d^2\bar{r}}{d\theta^2}=-\left[a\sin\theta\frac{dz}{d\theta}+a\cos\theta\frac{d^2z}{d\theta^2},\right.$$

$$\left.-a\sin\theta\frac{d^2z}{d\theta^2}+a\cos\theta\frac{dz}{d\theta};\ a^2\right]$$

$$\left|\frac{d\bar{r}}{d\theta}\times\frac{d^2\bar{r}}{d\theta^2}\right|=\left[\left(a\sin\theta\frac{dz}{d\theta}+a\cos\theta\frac{d^2z}{d\theta^2}\right)^2\right.$$

$$\left.+\left(a\cos\theta\frac{dz}{d\theta}-a\sin\theta\frac{d^2z}{d\theta^2}\right)^2+a^4\right]^{\frac{1}{2}}$$

$$=\left[a^2\left(\frac{dz}{d\theta}\right)^2+a^2\left(\frac{d^2z}{d\theta^2}\right)^2+a^4\right]^{\frac{1}{2}}$$

$$\text{and } \left|\frac{d\bar{r}}{d\theta}\right|=\left(a^2+\left(\frac{dz}{d\theta}\right)^2\right]^{\frac{1}{2}}$$

$$\therefore k=\frac{\left|\frac{d\bar{r}}{d\theta}\times\frac{d^2\bar{r}}{d\theta^2}\right|}{\left|\frac{d\bar{r}}{d\theta}\right|^3}$$

$$\therefore k^2=\frac{1}{\rho^2}=\frac{\left[a^2\left(\frac{dz}{d\theta}\right)^2+a^2\left(\frac{d^2z}{d\theta^2}\right)+a^4\right]}{\left[a^2+\left(\frac{dz}{d\theta}\right)^2\right]^3}$$

When a cylinder is developed into the plane, the point is given by

$$\bar{r}_1=\left(a(\pi+\theta),-a,z\right)$$

$$\frac{d\bar{r}_1}{d\theta}=\left(a,0,\frac{dz}{d\theta}\right)$$

$$\frac{d^2\bar{r}_1}{d\theta^2}=\left(0,0,\frac{d^2z}{d\theta^2}\right)$$

$$\therefore\frac{d\bar{r}_1}{d\theta}\times\frac{d^2\bar{r}_1}{d\theta^2}=\left(0,-a\frac{d^2z}{d\theta^2},0\right)$$

$$\therefore\left|\frac{d\bar{r}_1}{d\theta}\times\frac{d^2\bar{r}_1}{d\theta^2}\right|=\left[a^2\left(\frac{d^2z}{d\theta^2}\right)^2\right]^{1/2}$$

$$\text{and } \left|\frac{d\bar{r}_1}{d\theta}\right|=\left[a^2+\left(\frac{dz}{d\theta}\right)^2\right]^{1/2}$$

$$\therefore k_1^2=\frac{1}{\rho_1^2}=\frac{a^2\left(d^2\frac{z}{d}\theta^2\right)^2}{\left[a^2+\left(\frac{dz}{d}\theta\right)^2\right]^3}$$

$$\therefore\frac{1}{\rho^2}-\frac{1}{\rho_1^2}=\frac{a^2\left(\frac{dz}{d\theta}\right)^2+a^4}{\left[a^2+\left(\frac{dz}{d}\theta\right)^2\right]^3}$$

$$=\frac{a^2}{\left[a^2+\left(\frac{dz}{d\theta}\right)^2\right]^2}$$

If ϕ is the angle between the generator and the tangent to the curve

$$\cos\phi=\frac{k.\frac{d\bar{r}}{d\theta}}{\left|\frac{d\bar{r}}{d\theta}\right|}=\frac{\frac{dz}{d\theta}}{\left[a^2+\left(\frac{dz}{d\theta}\right)^2\right]^{1/2}}$$

$$\text{or } \left[a^2+\left(\frac{dz}{d\theta}\right)^2\right]\cos^2\phi=\left(\frac{dz}{d\theta}\right)^2$$

$$a^2\cos^2\phi=\left(\frac{dz}{d\theta}\right)^2\sin^2\phi;$$

$$\left(\frac{dz}{d\theta}\right)^2 = a^2 \cot^2 \phi$$

$$\therefore \frac{1}{\rho^2} - \frac{1}{\rho_1^2} = \frac{a^2}{a^4 \operatorname{cosec}^4 \phi} = \frac{\sin^4 \phi}{a^2}$$

Example 6

Find the curvature and torsion of the curve $x = a(3u - u^3)$; $y = 3au^2$; $z = a(3u + u^3)$

$$\bar{r} = a[3u - u^3, 3u^2, 3u + u^3]; \frac{d\bar{r}}{du} = 3a\,[(1-u^2), 2u, (1+u^2)]$$

$$\frac{d^2\bar{r}}{du^2} = 3a(-2u, 2, 2u); \frac{d^3\bar{r}}{du^3} = 6a(-1, 0, 1)$$

$$\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2} = 18a^2\,[u^2 - 1, -2u, 1 + u^2]$$

$$\left|\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right| = 18\sqrt{2}\,[u^2 + 1]$$

$$\therefore k = \frac{\left|\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right|}{\left|\frac{d\bar{r}_1}{du}\right|^3} = \frac{18\sqrt{2}a^2(1+u^2)}{54\sqrt{2}a^3(1+u^2)^3}$$

$$= \frac{1}{3a} \frac{1}{1+u^2}$$

$$\left[\frac{d\bar{r}}{du}, \frac{d^2\bar{r}}{du^2}, \frac{d^3\bar{r}}{du^3}\right] = \begin{vmatrix} 1-u^2 & 2u & 1+u^2 \\ -2u & 2 & 2u \\ -1 & 0 & 1 \end{vmatrix}$$

$$54a^3 = 4 \times 54a^3$$

$$\rho = \left[\frac{d\bar{r}}{du}, \frac{d^2\bar{r}}{du^2}, \frac{d^3\bar{r}}{du^3}\right] \div \left[\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right]^2$$

$$= \frac{4 \times 54a^3}{18 \times 18 \times 2a^4(1+u^2)^2} = \frac{1}{3a(1+u^2)^2}$$

Exercise: Calculate curvature and torsion for the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $z = b\theta$

Example 7

If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction, show that $\frac{\sin\theta\, d\theta}{\sin\phi\, d\phi} = -\frac{k}{\tau}$

where k and τ have their usual meaning

Let $\bar{d}$ be the fixed direction. Given $\bar{t}\cdot\bar{d} = d\cos\theta$ and $\bar{h}\cdot\bar{d} = d\cos\phi$ where $d = |\bar{d}|$.

Differentiating with respect to s,

$$\bar{t}'\cdot\bar{d} = -d\sin\theta\frac{d\theta}{ds}$$

$$\bar{t}'\cdot\bar{d} = -d\sin\phi\frac{d\phi}{ds} \quad (\because \bar{d} \text{ is a constant vector})$$

$$\therefore k\,\bar{n}\cdot\bar{d} = -d\sin\theta\frac{d\theta}{ds} \text{ and}$$

$$-\tau\,\bar{n}\cdot d = -d\sin\phi\frac{d\phi}{ds}$$ by Serret-Frenet formulae

Dividing, $-\frac{k}{\tau} = \frac{\sin\theta\, d\theta}{\sin\phi\, d\phi}$

2. STATICS

I. Fundamental ideas

Statics deals with forces (on bodies) which produce equilibrium.

If a single force $\vec{R}$ acting independently produces the same effect as that due to a number of forces F_1, F_2, F_3, etc., acting simultaneously, then $\bar{R}$ is called the **resultant** of $\vec{F_1}, \vec{F_2}, \vec{F_3}$, etc. In other words $\vec{R}$ is $\vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots$, The forces $\vec{F_1}, \vec{F_2}, \vec{F_3}$, etc., are called **components** of $\vec{R}$.

The resultant of two forces $\vec{P}$ and $\vec{Q}$ acting along OA and OB such that angle AOB is θ, is of magnitude $\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$. This is obtained by applying parallelogram law for addition of vectors.